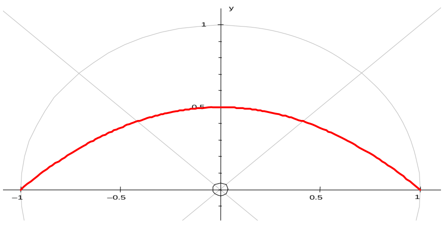
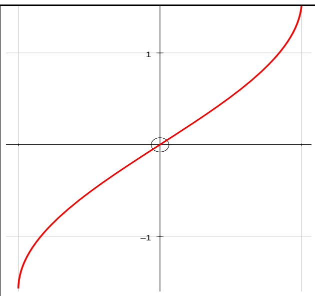
4756 (FP2) Further Methods for Advanced Mathematics

1 (a)(i)	$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots$ $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$ $= 2x + \frac{2x^3}{3} + \frac{2x^5}{5} \ldots$ Valid for $-1 < x < 1$	B1 M1 A1 B1 **4**	Series for $\ln(1-x)$ as far as x^5 s.o.i. Seeing series subtracted Inequalities must be strict
(ii)	$\frac{1+x}{1-x} = 3$ $\Rightarrow 1 + x = 3(1 - x)$ $\Rightarrow 1 + x = 3 - 3x$ $\Rightarrow 4x = 2$ $\Rightarrow x = \frac{1}{2}$ $\ln 3 \approx 2\times\frac{1}{2} + \frac{2}{3}\times\left(\frac{1}{2}\right)^3 + \frac{2}{5}\times\left(\frac{1}{2}\right)^5$ $= 1 + \frac{1}{12} + \frac{1}{80}$ $= 1.096$ (3 d.p.)	M1 A1 M1 A1 **4**	Correct method of solution B2 for $x = ½$ stated Substituting their x into their series in (a) (i), even if outside range of validity. Series must have at least two terms *SR*: if >3 correct terms seen in (i), allow a better answer to 3 d.p. Must be 3 decimal places
(b)(i)		G1 G1 G1 **3**	$r(0) = a$, $r(\pi/2) = a/2$ indicated Symmetry in $\theta = \pi/2$ Correct basic shape: flat at $\theta = \pi/2$, not vertical or horizontal at ends, no dimple Ignore beyond $0 \le \theta \le \pi$
(ii)	$r + y = r + r\sin\theta$ $= r(1 + \sin\theta) = \frac{a}{1+\sin\theta} \times (1 + \sin\theta)$ $= a$ $\Rightarrow r = a - y$ $\Rightarrow x^2 + y^2 = (a - y)^2$ $\Rightarrow x^2 + y^2 = a^2 - 2ay + y^2$ $\Rightarrow 2ay = a^2 - x^2$ $\Rightarrow y = \frac{a^2 - x^2}{2a}$	M1 A1 (ag) M1 A1 A1 **5**	Using $y = r\sin\theta$ Using $r^2 = x^2 + y^2$ in $r + y = a$ Unsimplified A correct final answer, not spoiled **16**

2 (i)	$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 1 & -2\\ 0 & -1-\lambda & 0\\ 2 & 0 & 1-\lambda\end{pmatrix}$		
	det($\mathbf{M}$ – $\lambda\mathbf{I}$) = $(3-\lambda)[(-1-\lambda)(1-\lambda)] + 2[2(-1-\lambda)]$	M1	Attempt at det($\mathbf{M}$ – $\lambda\mathbf{I}$) with all elements present. Allow sign errors
	$= (3-\lambda)(\lambda^2-1) + 4(-1-\lambda)$	A1	Unsimplified. Allow signs reversed. Condone omission of = 0
	$\Rightarrow$ $\lambda^3 - 3\lambda^2 + 3\lambda + 7 = 0$ det $\mathbf{M}$ = –7	B1 **3**	
(ii)	$f(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda + 7$		
	$f(-1) = -1 - 3 - 3 + 7 = 0 \Rightarrow -1$ eigenvalue	B1	Showing –1 satisfies a correct characteristic equation
	$f(\lambda) = (\lambda + 1)(\lambda^2 - 4\lambda + 7)$ $\lambda^2 - 4\lambda + 7 = (\lambda - 2)^2 + 3 \geq 3$ so no real roots $(\mathbf{M}-\lambda\mathbf{I})\mathbf{s} = \mathbf{0}$, $\lambda = -1$	M1 A1	Obtaining quadratic factor www $(\mathbf{M}-\lambda\mathbf{I})\mathbf{s} = (\lambda)\mathbf{s}$ M0 below
	$\Rightarrow \begin{pmatrix}4 & 1 & -2\\ 0 & 0 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$		
	$\Rightarrow$ $4x + y - 2z = 0$ $2x + 2z = 0$	M1	Obtaining equations relating x, y and z
	$\Rightarrow$ $x = -z$ $y = 2z - 4x = 2z + 4z = 6z$	M1	Obtaining equations relating two variables to a third. Dep. on first M1
	$\Rightarrow$ $\mathbf{s} = \begin{pmatrix}-1\\ 6\\ 1\end{pmatrix}$	A1	Or any non-zero multiple
	$\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-0.1\\ 0.6\\ 0.1\end{pmatrix}$	M1	Solution by any method, e.g. use of multiple of $\mathbf{s}$, but M0 if $\mathbf{s}$ itself quoted without further work
	$\Rightarrow$ $x = 0.1$, $y = -0.6$, $z = -0.1$	A2 **9**	Give A1 if any two correct
(iii)	C-H: a matrix satisfies its own characteristic equation $\Rightarrow$ $\mathbf{M}^3 - 3\mathbf{M}^2 + 3\mathbf{M} + 7\mathbf{I} = \mathbf{0}$	B1	Idea of $\lambda \leftrightarrow \mathbf{M}$
	$\Rightarrow$ $\mathbf{M}^3 = 3\mathbf{M}^2 - 3\mathbf{M} - 7\mathbf{I}$	B1 (ag)	Must be derived www. Condone omitted $\mathbf{I}$
	$\Rightarrow$ $\mathbf{M}^2 = 3\mathbf{M} - 3\mathbf{I} - 7\mathbf{M}^{-1}$	M1	Multiplying by $\mathbf{M}^{-1}$
	$\Rightarrow$ $\mathbf{M}^{-1} = -\frac{1}{7}\mathbf{M}^2 + \frac{3}{7}\mathbf{M} - \frac{3}{7}\mathbf{I}$	A1 **4**	o.e.
(iv)	$\mathbf{M}^2 = \begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}=\begin{pmatrix}5 & 2 & -8\\ 0 & 1 & 0\\ 8 & 2 & -3\end{pmatrix}$	M1	Correct attempt to find $\mathbf{M}^2$
	$-\frac{1}{7}\begin{pmatrix}5 & 2 & -8\\ 0 & 1 & 0\\ 8 & 2 & -3\end{pmatrix}+\frac{3}{7}\begin{pmatrix}3 & 1 & -2\\ 0 & -1 & 0\\ 2 & 0 & 1\end{pmatrix}-\frac{3}{7}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	M1	Using their (iii)
	$=\begin{pmatrix}\frac{1}{7} & \frac{1}{7} & \frac{2}{7}\\ 0 & -1 & 0\\ -\frac{2}{7} & -\frac{2}{7} & \frac{3}{7}\end{pmatrix}$ or $\frac{1}{7}\begin{pmatrix}1 & 1 & 2\\ 0 & -7 & 0\\ -2 & -2 & 3\end{pmatrix}$	A1	SC1 for answer without working

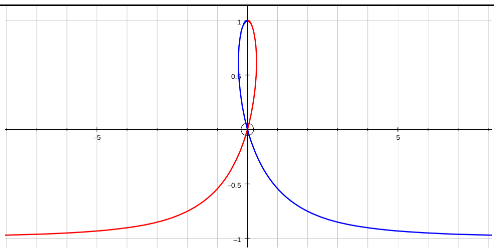
	OR Matrix of cofactors: $\begin{pmatrix} -1 & 0 & 2 \\ -1 & 7 & 2 \\ -2 & 0 & -3 \end{pmatrix}$	M1		Finding at least four cofactors
	Adjugate matrix $\begin{pmatrix} -1 & -1 & -2 \\ 0 & 7 & 0 \\ 2 & 2 & -3 \end{pmatrix}$: det **M** = –7	M1	**3**	Transposing and dividing by determinant. Dep. on M1 above **19**

3(a)(i)		G1	**1**	Correct basic shape (positive gradient, through (0, 0))
	$y = \arcsin x \Rightarrow \sin y = x$	M1		$\sin y =$ and attempt to diff. both sides
	$\Rightarrow \dfrac{dx}{dy} = \cos y$	A1		Or $\cos y \dfrac{dy}{dx} = 1$
	$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1-x^2}}$	A1		www. SC1 if quoted without working
	Positive square root because gradient positive	B1	**4**	Dep. on graph of an increasing function
(ii)	$\int_0^1 \dfrac{1}{\sqrt{2-x^2}}\,dx = \left[\arcsin \dfrac{x}{\sqrt{2}}\right]_0^1$	M1		arcsin function alone, or any sine substitution
		A1		$\dfrac{x}{\sqrt{2}}$, or $\int 1\,d\theta$ www without limits
	$= \dfrac{\pi}{4}$	A1	**3**	Evaluated in terms of π
(b)	$C + jS = e^{j\theta} + \frac{1}{3}e^{3j\theta} + \frac{1}{9}e^{5j\theta} + \ldots$	M1		Forming $C + jS$ as a series of powers
	This is a geometric series	M1		Identifying geometric series and attempting sum to infinity or to n terms
	with first term $a = e^{j\theta}$, common ratio $r = \frac{1}{3}e^{2j\theta}$	A1		Correct a and r
	Sum to infinity $= \dfrac{a}{1-r} = \dfrac{e^{j\theta}}{1-\frac{1}{3}e^{2j\theta}}$ $\left(= \dfrac{3e^{j\theta}}{3-e^{2j\theta}}\right)$	A1		Sum to infinity
	$= \dfrac{3e^{j\theta}}{3-e^{2j\theta}} \times \dfrac{3-e^{-2j\theta}}{3-e^{-2j\theta}}$	M1*		Multiplying numerator and denominator by $1-\frac{1}{3}e^{-2j\theta}$ o.e. Or writing in terms of trig functions and realising the denominator
	$= \dfrac{9e^{j\theta} - 3e^{-j\theta}}{9 - 3e^{-2j\theta} - 3e^{2j\theta} + 1}$	M1		Multiplying out numerator and denominator. Dep. on M1*
	$= \dfrac{9(\cos\theta + j\sin\theta) - 3(\cos\theta - j\sin\theta)}{10 - 3(\cos 2\theta - j\sin 2\theta) - 3(\cos 2\theta + j\sin 2\theta)}$	M1		Valid attempt to express in terms of trig functions. If trig functions used from start, M1 for using the compound angle formulae and Pythagoras Dep. on M1*
	$= \dfrac{6\cos\theta + 12j\sin\theta}{10 - 6\cos 2\theta}$	A1		
	$\Rightarrow C = \dfrac{6\cos\theta}{10 - 6\cos 2\theta}$	M1		Equating real and imaginary parts. Dep. on M1*

	$= \dfrac{3\cos\theta}{5-3\cos 2\theta}$	A1 (ag)	
	$S = \dfrac{6\sin\theta}{5-3\cos 2\theta}$	A1	o.e.
		11	**19**

4 (i)	cosh $u = \frac{e^u + e^{-u}}{2}$ $\Rightarrow$ 2 cosh$^2u = \frac{e^{2u} + 2 + e^{-2u}}{2}$ $\Rightarrow$ 2 cosh$^2u - 1 = \frac{e^{2u} + e^{-2u}}{2}$ = cosh 2u	 B1 B1 B1 (ag) **3**	 $\left(e^u + e^{-u}\right)^2 = e^{2u} + 2 + e^{-2u}$ cosh 2u = $\frac{e^{2u} + e^{-2u}}{2}$ Completion www
(ii)	x = arsinh y $\Rightarrow$ sinh $x = y$ $\Rightarrow$ $y = \frac{e^x - e^{-x}}{2}$ $\Rightarrow$ $e^{2x} - 2ye^x - 1 = 0$ $\Rightarrow$ $(e^x - y)^2 - y^2 - 1 = 0$ $\Rightarrow$ $(e^x - y)^2 = y^2 + 1$ $\Rightarrow$ $e^x - y = \pm\sqrt{y^2+1}$ $\Rightarrow$ $e^x = y \pm \sqrt{y^2+1}$ Take + because $e^x > 0$ $\Rightarrow$ $x = \ln(y + \sqrt{y^2+1})$	M1 M1 B1 A1 (ag) **4**	Expressing y in exponential form ($\frac{1}{2}$, – must be correct) Reaching e^x by quadratic formula or completing the square. Condone no ± Or argument of ln must be positive Completion www but independent of B1
(iii)	$x = 2\sinh u \Rightarrow \frac{dx}{du} = 2\cosh u$ $\int \sqrt{x^2+4}\, dx = \int \sqrt{4\sinh^2 u + 4} \times 2\cosh u\, du$ $= \int 4\cosh^2 u\, du$ $= \int 2\cosh 2u + 2\, du$ = sinh 2u + 2u + c = 2 sinh u cosh u + 2u + c $= x\sqrt{1+\frac{x^2}{4}}$ + 2 arsinh $\frac{x}{2}$ + c $= \frac{1}{2}x\sqrt{4+x^2}$ + 2 arsinh $\frac{x}{2}$ + c	M1 A1 M1 A1 M1 A1 (ag) **6**	$\frac{dx}{du}$ and substituting for all elements Substituting for all elements correctly Simplifying to an integrable form Any form, e.g. $\frac{1}{2}e^{2u} - \frac{1}{2}e^{-2u} + 2u$ Condone omission of + c throughout Using double "angle" formula and attempt to express cosh u in terms of x Completion www
(iv)	$t^2 + 2t + 5 = (t+1)^2 + 4$ $\int_{-1}^{1} \sqrt{t^2+2t+5}\, dt = \int_{-1}^{1} \sqrt{(t+1)^2+4}\, dt$ $= \int_0^2 \sqrt{x^2+4}\, dx$ $= \left[\frac{1}{2}x\sqrt{4+x^2} + 2\,\text{arsinh}\,\frac{x}{2}\right]_0^2$	B1 M1 A1	Completing the square Simplifying to an integrable form, by substituting $x = t + 1$ s.o.i. or complete alternative method Correct limits consistent with their method seen anywhere

	$= \sqrt{8} + 2$ arsinh 1 $= 2\sqrt{2} + 2\ln(1 + \sqrt{2})$ $= 2(\ln(1 + \sqrt{2}) + \sqrt{2})$	M1 A1 (ag) **5**	Using (iii) or otherwise reaching the result of integration, and using limits Completion www. Condone $\sqrt{8}$ etc. **18**

5 (i)	If $a = 1$, angle OCP = 45° so P is (1 – cos 45°, sin 45°) $\Rightarrow$ P $\left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	 M1 A1 (ag)	 Completion www
	OR Circle $(x-1)^2 + y^2 = 1$, line $y = -x + 1$ $(x-1)^2 + (-x+1)^2 = 1$ M1 $\Rightarrow$ $x = 1 \pm \frac{1}{\sqrt{2}}$ and hence P A1		Complete algebraic method to find x
	Q $\left(1 + \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	B1 **3**	
(ii)	cos OCP = $\frac{a}{\sqrt{a^2+1}}$ sin OCP = $\frac{1}{\sqrt{a^2+1}}$ P is ($a - a$ cos OCP, a sin OCP) $\Rightarrow$ P $\left(a - \frac{a^2}{\sqrt{a^2+1}}, \frac{a}{\sqrt{a^2+1}}\right)$	M1 A1 A1 (ag)	Attempt to find cos OCP and sin OCP in terms of a Both correct Completion www
	OR Circle $(x-a)^2 + y^2 = a^2$, line $y = -\frac{1}{a}x + 1$ $(x-a)^2 + \left(-\frac{1}{a}x+1\right)^2 = a^2$ M1 $\Rightarrow$ $x = \dfrac{2a + \frac{2}{a} \pm \sqrt{\left(2a+\frac{2}{a}\right)^2 - 4\left(1+\frac{1}{a^2}\right)}}{2\left(1+\frac{1}{a^2}\right)}$ A1 $\Rightarrow$ $x = a \pm \frac{a^2}{\sqrt{a^2+1}}$ and hence P A1		 Complete algebraic method to find x Unsimplified
	Q $\left(a + \frac{a^2}{\sqrt{a^2+1}}, -\frac{a}{\sqrt{a^2+1}}\right)$	B1 **4**	
(iii)	 As $a \to \infty$, P $\to$ (0, 1) As $a \to -\infty$, y co-ordinate of P $\to -1$ $\frac{a}{\sqrt{a^2+1}} \to \frac{a}{-a} = -1$ as $a \to -\infty$	G1 G1 G1 G1ft B1 B1 M1 A1 **8**	Locus of P (1st & 3rd quadrants) through (0, 0) Locus of P terminates at (0, 1) Locus of P: fully correct shape Locus of Q (2nd & 4th quadrants: dotted) reflection of locus of P in y-axis Stated separately Stated Attempt to consider y as $a \to -\infty$ Completion www
(iv)	POQ = 90° Angle in semicircle Loci cross at 90°	B1 B1 B1 **3**	 o.e. **18**